# Fighting Opinion Control in Social Networks via Link Recommendation

Victor Amelkin (University of Pennsylvania)  
Ambuj K. Singh (University of California, Santa Barbara)

## Introduction

- **Goal:** Strategically recommend links to recover weighted average user opinion from exogenous node-level attacks.  
- Large directed strongly connected social network of $n$ users  
- $W$ - interpersonal influence adjacency matrix ($W\mathbf{1}=1$)  
- $x\in[0,1]^n$, $(x,\bar{x})$ - user opinions before (after) the attack  
- $\pi\in\mathbb{R}^n$ - network nodes’ eigenvector centralities  
- $\langle\pi,x\rangle$ - (weighted) average opinion

## Problem Statement by Example

<table>
<thead>
<tr>
<th>Original network $W$ and user opinion $\pi$</th>
<th>Adversary changes $x$ (goal: $(\pi,\bar{x}) \rightarrow \max$)</th>
<th>Adding edges to network $W$ (goal: $(\pi,\bar{x}) \rightarrow (\bar{\pi},x)$)</th>
</tr>
</thead>
</table>
| $\pi = \{0.06, 0.01, 0.09, 0.21\}^T$  
$\bar{x} = \{0.3, 0.1, 0.2, 0.06\}^T$  
$(\pi,\bar{x}) = 0.35$ |
| $\bar{W} = \{0.06, 0.01, 0.09, 0.21\}^T$  
$\bar{x} = \{0.3, 0.1, 0.2, 0.1\}^T$  
$(\bar{\pi},x) = 0.43$ |
| $\bar{W} = \{0.46, 0.18, 0.20, 0.14\}^T$  
$\bar{x} = \{0.3, 0.1, 0.2, 0.1\}^T$  
$(\bar{\pi},x) = 0.36 \approx (\pi,\bar{x})$ |

## Problem and Its Hardness

- **Adversary’s Goal:** Maximize $\langle\pi,\bar{x}\rangle$ via altering $x \rightarrow \bar{x}$  
- **Our Goal:** Return $\langle\bar{\pi},x\rangle$ back to $\langle\pi,x\rangle$ via altering $\pi \rightarrow \bar{\pi}$ through edge addition. Single-edge $(r,c)$ perturbation:  

$$\pi_j = \pi_j \left(1 - \theta_{rc}\delta(r,c)\right)$$

- **(NP-hard) Problem:**

$$\text{DIVER}(W,k,x,\bar{x}) = \arg\min_{\bar{W}} |\langle\bar{\pi}(W),\bar{x}\rangle - \langle\pi,x\rangle|$$

where the perturbed $\bar{W}$ differs from $W$ by $k$ new edges, we cannot choose weight $\theta_{ij}$ of an added edge $(i,j)$.

## General Solution for DIVER

$$\text{DIVER}(W,k,x,\bar{x}) = \arg\min_{\bar{W}} |\langle\bar{\pi}(W),\bar{x}\rangle - \langle\pi,x\rangle|$$

- **Method:** reduce $\langle\bar{\pi},x\rangle$ through iterative edge addition until it gets close enough to $\langle\pi,x\rangle$  

$$\langle\bar{\pi},x\rangle - \langle\bar{\pi},\bar{x}\rangle \rightarrow \max$$

- **Central Question:** How does $\langle\bar{\pi},x\rangle$ change when a single edge $(r,c)$ with weight $\theta_{rc}$ is added to network $W$?

## Network Perturbation Analysis

- Adding a single edge to the network:

$$\bar{W} = W - \theta_{rc}\delta(r,c)W + \theta_{rc}e_re_c^T$$

**Theorem 1.** Under single-edge perturbation of $W$ with edge $(r,c)$ having weight $\theta_{rc}$, the eigenvector centrality changes as follows:

$$\bar{\pi}_j = \pi_j \left[1 - \frac{1}{m_{rr} + \theta_{rc}(m_{rc} - m_{rr} + 1)} \right],$$

where $m_{ij}$ is the mean first passage time (MFPT) from state $i$ to state $j$ of Markov chain $W$, and $\delta\{i,j\}$ is Kronecker’s delta. In particular,

$$\bar{\pi}_r = \frac{1}{m_{rr} + \theta_{rc}(m_{rc} - m_{rr} + 1)},$$

$$\bar{\pi}_c = 1 + \theta_{rc} \cdot \frac{m_{rc} - 1}{m_{rr} + \theta_{rc}(m_{rc} - m_{rr} + 1)}.$$  

**Theorem 2.** Under single-edge perturbation of $W$ with edge $(r,c)$ having weight $\theta_{rc}$, the weighted average opinion changes as follows:

$$f_x(r,c) = \langle\pi,\bar{x}\rangle - \langle\bar{\pi},\bar{x}\rangle = \sum_{j=1}^n \pi_j m_{cj} \left(1 - \delta\{j,c\}\right) - m_{cj} + 1)\bar{x}_j$$

$$+ \theta_{rc} \sum_{j=1}^n m_{rc} \left(1 - \delta\{j,r\}\right) - m_{rc} + 1)\bar{x}_r.$$  

## Efficient computation of $f_x(r,c)$

- In hierarchical networks, $f_x$ is largely determined by a small $(n_{src})$ number of top-centrality nodes  
- We can estimate MFPTs via finite-time random walks; all the MFPTs to and from $n_{src}$ top-centrality nodes converge in $O(n)$ time in practice.

**Outcome:** $O(n)$-time heuristic for DIVER for hierarchical (scale-free-like) networks.

## Efficient candidate edge selection

- Focus on $O(n)$ candidate edges, outgoing from $n_{src} \ll n$ nodes.  
- Most good candidate edges emanate from a small number of nodes.  
- In hierarchical networks, these edge sources are top-centrality nodes.

## References

- Amelkin V., Singh A.K., “Fighting Opinion Control in Social Networks via Link Recommendation”, in Proc. of ACM SIGKDD, 2019

---

The work was supported by the U.S. Army Research Laboratory and the U.S. Army Research Office under grant W911NF-15-1-0577, by the National Science Foundation under grant IIS-1817046, and the Rockefeller Foundation under grant 2017 PRE 301.

https://victoramelkin.com/pub/diver/vctv@seas.upenn.edu; ambuj@cs.ucsb.edu