Contagion and Equilibria in Diversified Financial Networks

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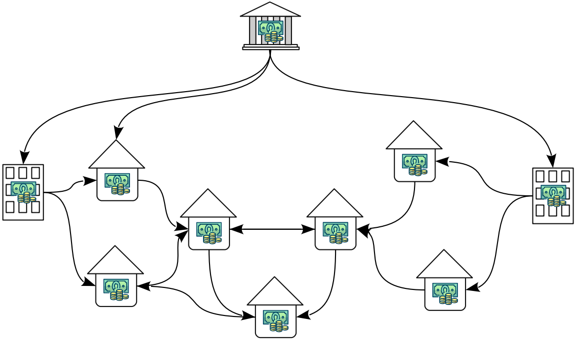
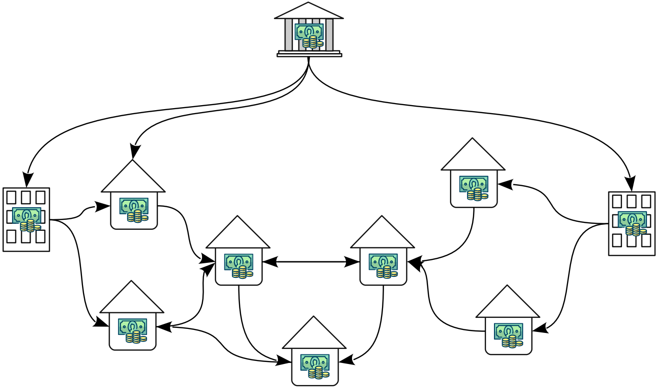
Joint work with Santosh Venkatesh and Rakesh Vohra (both at Penn)

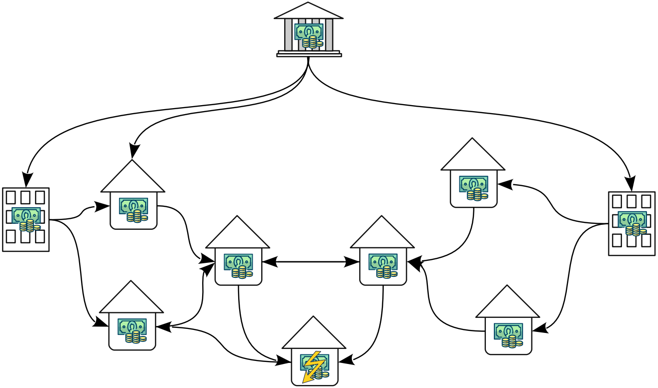


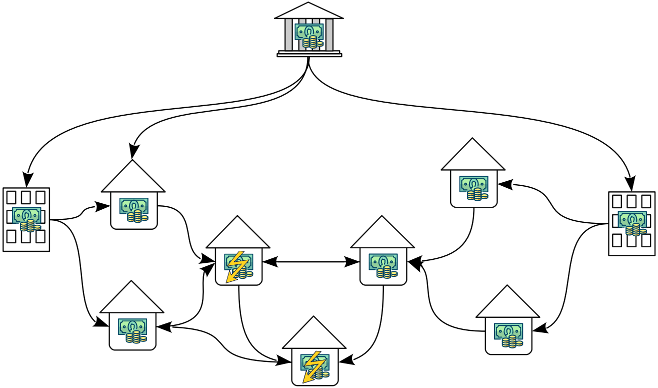
- A network of firms
- Firms have cash endowments
- Firms own equity shares of other firms
- Cash endowments and equity ownership determine firms' market values

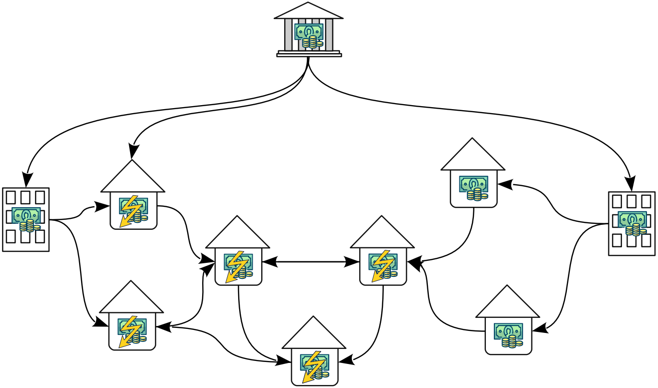
Financial Networks and Systemic Risk

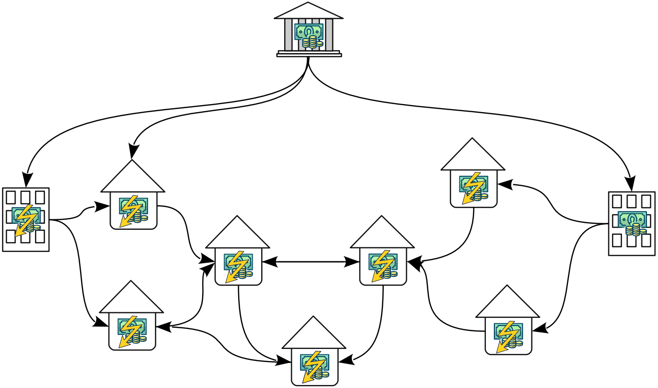


Financial Networks and Systemic Risk









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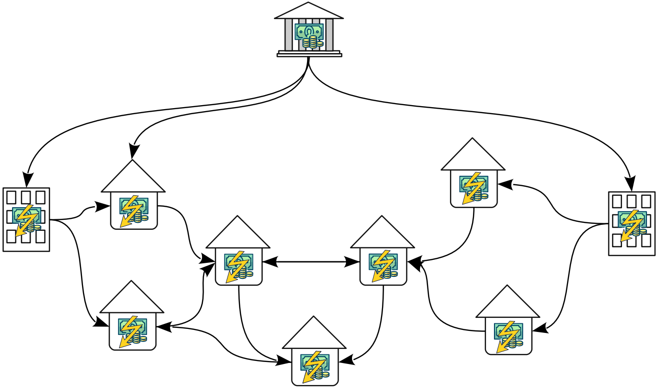
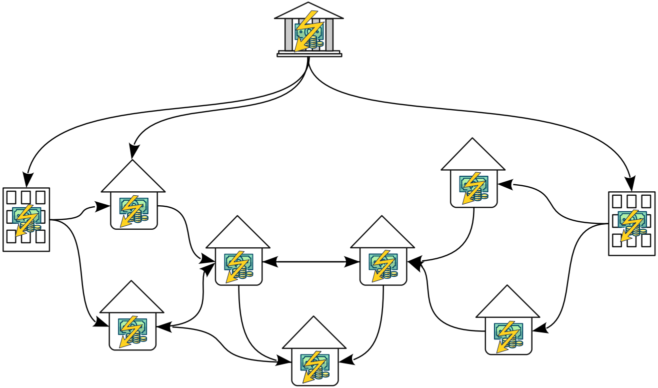
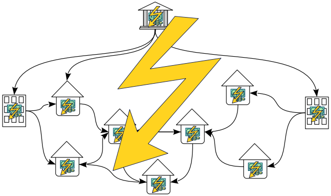




Financial Networks and Systemic Risk



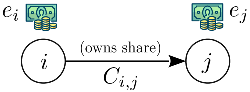
- Network was unable to absorb (one) failure
- **Key Question:** Given cash reserves and network, is there a risk of such economy collapse (*systemic risk*)?

Model

- $\boldsymbol{V}(t) \in \mathbb{R}^n$ – firms' market values at time t
- $\boldsymbol{e} \in \mathbb{R}^n$ – firms' cash endowments
- $\boldsymbol{C}$ – adjacency matrix of (**fixed** or **random**) network; $(\boldsymbol{C}^\intercal \mathbb{1})_i = c \in (0,1)$
  $(C_{ij} \in (0,1)$ – relative share of firm j owned by firm i)
- $\mathbb{1}_{\{\boldsymbol{V}(t) \leq \tau\}}$ – 1-0-indicator vector
- β, τ – fixed scalars

$$\boldsymbol{V} = \boldsymbol{e} + \boldsymbol{C}\boldsymbol{V} - \beta \mathbb{1}_{\{\boldsymbol{V} \leq \tau\}} \quad \text{(equilibrium form)}$$

$$\frac{d\boldsymbol{V}(t)}{dt} = \boldsymbol{e} - (\boldsymbol{I} - \boldsymbol{C})\boldsymbol{V}(t) - \beta \mathbb{1}_{\{\boldsymbol{V}(t) \leq \tau\}} \quad \text{(dynamic form)}$$

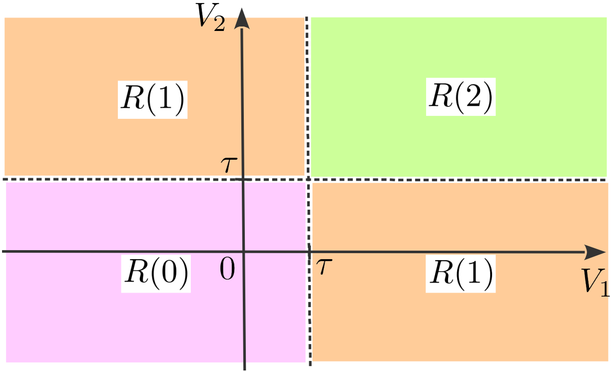


- **def:** if $V_i(t) \leq \tau$, then i is *distressed* or *insolvent*; otherwise, it is *solvent*
- **def:** *economy collapse* = a large number (e.g., n) of firms are insolvent

Model – State Space

- $\boldsymbol{V}(t) \in \cup_{k=1}^{n} R(k) \subset \mathbb{R}^n$

def: A *region* $R(k)$ is a set of $\boldsymbol{V}$ with k solvent firms ($V_i > \tau$). Regions correspond to orthants when the origin is moved to $(\tau, \dots, \tau)$.



$$\boldsymbol{V} = \boldsymbol{e} + \boldsymbol{C}\boldsymbol{V} - \beta\mathbb{1}_{\{\boldsymbol{V} \leq \tau\}}$$

$$\frac{d\boldsymbol{V}(t)}{dt} = \boldsymbol{e} - (\boldsymbol{I} - \boldsymbol{C})\boldsymbol{V}(t) - \beta\mathbb{1}_{\{\boldsymbol{V}(t) \leq \tau\}}$$

- **Q1: What are the model's equilibria?**
  - $\boldsymbol{V}^*$ – consistent combination of firm market values
- **Q2: What is the link between equilibria of random network instances?**
- **Q3: What is the model's dynamics around these equilibria?**
  - *actual adjustment* of firm market values $\boldsymbol{V}(t)$ in time starting at $\boldsymbol{V}(0)$ as result of dividend cross-payment + shock absorption
  - *fictitious dynamics* seeking true market values starting from an estimate using local information

Theorem (About Equilibrium Point Distribution in Regular Cliques)

In model with homogeneous cash endowments $\boldsymbol{e} = e\mathbb{1}$, for fixed regular clique $\boldsymbol{C} = \frac{c}{n}\mathbb{1}\mathbb{1}^\intercal$, $(\boldsymbol{C}^\intercal\mathbb{1})_i = c$, region $R(k)$ with k solvent firms has the unique equilibrium point

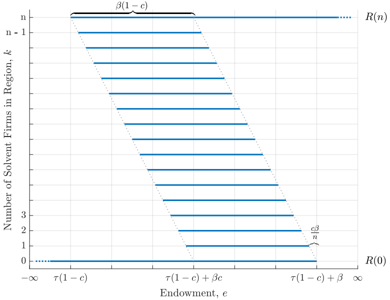
$$V_i^{*R(k)} = \frac{e - c\beta\frac{n-k}{n}}{1-c} - \beta\delta\{\textit{firm } i \textit{ is insolvent in } R(k)\}$$

if and only if cash endowment e is in range

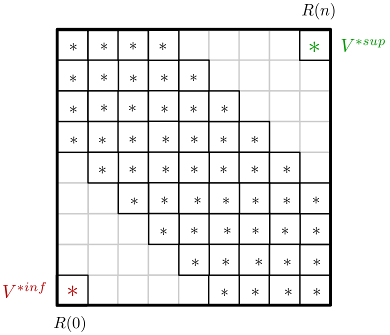
$$e \in \left(\tau(1-c) + c\beta\frac{n-k}{n}; (\tau+\beta)(1-c) + c\beta\frac{n-k}{n}\right]$$

except for the region $R(0)$ for which the lower bound on e is $-\infty$, and region $R(n)$ for which the upper bound on e is $+\infty$.

- Regions $R(k)$ have equilibria when e falls within their ranges.

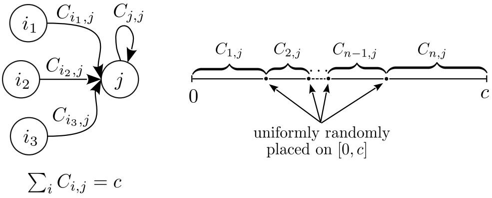


- Equilibria distribution for "mid-range" e



(geometrically imperfect diagram; cells – orthants; "distance" to V^{*sup}'s orthant $\sim$ number of solvent firms)

- Results apply to *exchangeable systems* with ≤ 8-order moments "not too large" (no "dominant" investor(s) owning the bulk of equity)
- This talk – de Finetti: each firm's equity c is uniformly randomly split among everyone



- $\boldsymbol{C}$'s columns are independent; entries of one column are correlated

$$\mathbb{P}\left\{C_{1j} > x_1, \ldots, C_{nj} > x_n\right\} = \max\left\{0, \left(1 - \tfrac{x_1}{c} - \cdots - \tfrac{x_n}{c}\right)^{n-1}\right\}$$

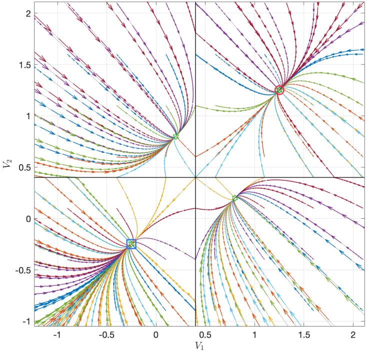
$$\mathbb{E}(C_{1j}) = \frac{c}{n}, \quad \mathrm{Var}(C_{1j}) = \frac{(n-1)c^2}{(n+1)n^2}, \quad \mathrm{Cov}(C_{1j}, C_{2j}) = -\frac{c^2}{(n+1)n^2}$$

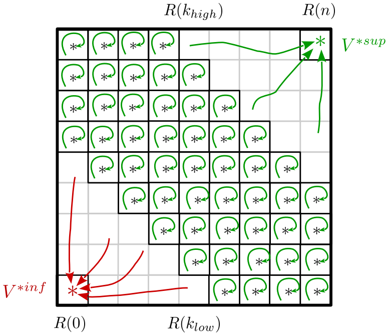
- Networks are related to cliques; but locally – highly irregular

Theorem ((Simplified) Equilibrium Distribution in Random Cliques)

In the homogeneous endowment model, when network size n *gets sufficiently large, equilibria* $\boldsymbol{V}$ *of an instance of a random clique get arbitrarily close* ($\sup$ *over vector components) to the corresponding equilibria* $\boldsymbol{V}^{\circ}$ *of the regular clique with high probability under mild assumptions on the network distribution.*

$$\frac{d\boldsymbol{V}(t)}{dt} = \boldsymbol{e} - (\boldsymbol{I} - \boldsymbol{C})\boldsymbol{V}(t) - \beta \mathbb{1}_{\{\boldsymbol{V}(t) \leq \tau\}}$$





Summary and Open Problems

1. Characterized the whole set of equilibria for a wide class of networks
2. Random network equilibria concentrate on equilibria of the regular clique
3. Regularity – property of the distribution rather than one network instance
4. Results on dynamics help analyze the impact of shocks
   - What if the system is kicked out of the all-firms-solvent state?

Open Questions:

1. Will we observe similar concentration in more heterogeneous networks?
   - same model over Erdos-Renyi (ER) topology – yes
   - some variance in investor scale (2 firms groups, one gets somewhat higher equity shares) – likely yes
   - heterogeneous endowments – likely yes
   - beyond ER – hard to characterize the limit for concentration
2. Will the same result apply to heterogeneous $\boldsymbol{c} \neq c\mathbb{1}$? – likely yes
3. Would similar results hold for other piece-wise linear models?
4. Would the same model apply to strongly disconnected networks (e.g., core-periphery)? – no, need a different model (with limited liability)

$\sim$ Questions $\sim$

Plan to distribute paper draft in $\sim$ 2 weeks;
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